JOURNAL OF DIFFERENTIAL EQUATIONS 84, 252-272 (1990)

Singular Solutions of Elliptic Equations and the Determination of Conductivity by Boundary Measurements*

GIOVANNI ALESSANDRINI

Istituto di Matematica, Facoltà di Ingegneria,
Università di Ancona,
Via delle Brecce Bianche, Ancona, Italy,

Received November 14, 1988

We improve some results on uniqueness and stability in the determination of the coefficient a in the equation (i) $\operatorname{div}(a \operatorname{grad} u) = 0$ in Ω, when all possible pairs of Dirichlet and Neumann data on $\partial\Omega$ are known. We also treat cases of anisotropic equations. Our method relies on the construction of solutions to (i) having an isolated singularity with prescribed asymptotic behaviour.

INTRODUCTION

In this paper we deal with the inverse problem of determining the coefficient $a = a(x) > 0$ in the elliptic equation $\operatorname{div}(a \operatorname{grad} u) = 0$ in Ω when the so-called Dirichlet to Neumann operator $\Lambda_a\colon u|_{\partial\Omega} \to a(\partial/\partial\nu)\, u|_{\partial\Omega}$ is given. Here Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and ν is the exterior normal to $\partial\Omega$.

We will mainly treat questions of uniqueness and stability (that is, continuous dependence on the data) for the boundary values of a and of its derivatives. Results of this kind are useful to infer uniqueness and stability for the interior values of a.

Kohn and Vogelius [KV1, KV2] proved that if $\partial\Omega \in C^\infty$ and a is piecewise analytic in Ω then Λ_a uniquely determines a. Sylvester and Uhlmann [SU1] proved that if $n \geqslant 3$ and $\partial\Omega \in C^\infty$, then Λ_a uniquely determines a in $C^\infty(\bar{\Omega})$.

Here, as by-products of our results, we obtain that, if $\partial\Omega$ and a have Lipschitz regularity, then Λ_a uniquely determines a among all the piecewise analytic perturbations of a (that is, all the functions $b = a + \varphi$ with piecewise analytic φ, see Corollary 1.1).

* Work partially supported by Fondi MPI (40%).

0022-0396/90 $3.00

Moreover, exploiting results due to Nachman, Sylvester, and Uhlmann [NSU], we obtain that, if $n \geqslant 3$, then Λ_a uniquely determines a in $W^{2,\infty}(\Omega)$, the class of functions having bounded second order generalized derivatives (see Corollary 1.2).

Our main result can be stated as follows (Theorem 1.3).

(I) *Let $a, b \in \mathrm{Lip}(\bar{\Omega})$, if $b-a$ is C^k in a neighborhood of $\partial\Omega$ then $\Lambda_a = \Lambda_b$ implies $D^\eta a = D^\eta b$ on $\partial\Omega$, for every multiindex η, $|\eta| \leqslant k$.*

Moreover we have the following stability estimate (Theorem 1.2).

(II) *If $a, b \in \mathrm{Lip}(\bar{\Omega})$ and $b-a$ is $C^{k+\alpha}$ in a neighborhood of $\partial\Omega$ then*

(a) $$\|D^\eta(a-b)\|_{L^\infty(\partial\Omega)} \leqslant \mathrm{Const}\, \|\Lambda_a - \Lambda_b\|^{\delta_k}_{L(H^{1/2}, H^{-1/2})}, \qquad \delta_k = \prod_{j=0}^{k} \frac{\alpha}{\alpha+j},$$

for every multiindex η, $|\eta| \leqslant k$.

Here the norm acting on $\Lambda_a - \Lambda_b$ is the operator norm for linear operators: $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

In Theorem 1.4 we extend the above results (I) and (II) to certain anisotropic equations: $\mathrm{div}(A \operatorname{grad} u) = 0$. It is well known that, in general, the Dirichlet to Neumann map $\Lambda_A : u|_{\partial\Omega} \to (ADu \cdot \nu)|_{\partial\Omega}$ cannot determine uniquely the matrix of coefficients A. Here we adopt a restricted viewpoint: we seek A of the form $A(a(x))$, where $A = A(t)$ is a differentiable one parameter family of symmetric positive definite matrices, and a is a scalar parameter depending on the space variable x. One extra assumption seems necessary in our method: the function $t \to A(t)$ must be monotone, that is, for every t, $(d/dt)\, A(t)$ must be a positive definite matrix.

Let us notice that, in the isotropic case, (I) was proven with C^∞-smooth a, b and $\partial\Omega$ by Kohn and Vogelius [KV1]. The estimate (a) with $|\eta| = 0$ was proven by Sylvester and Uhlmann [SU2], in fact their proof holds for coefficients which are just continuous. In case $|\eta| = 1$, an estimate like (a) was proven in [A], but a higher regularity on a, b was required.

Our argument can be sketched as follows. Let $\langle \cdot, \cdot \rangle$ be the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Then, using the definition of the Dirichlet to Neumann operator and by its selfadjointness, for every $u, v \in H^1(\Omega)$ satisfying, in the weak sense, $\mathrm{div}(a \operatorname{grad} u) = 0$, $\mathrm{div}(b \operatorname{grad} v) = 0$ in Ω, we obtain

(b) $$\int_\Omega (a-b)\, Du \cdot Dv = \langle (\Lambda_a - \Lambda_b)\, u, v \rangle$$

(see [A, Lemma 1] for an analogous formula).

Let z_σ be any point at distance $O(\sigma)$ from $\bar{\Omega}$, with σ small, and let us continue a, b in a neighborhood of Ω, containing z_σ. We will construct solutions u, v having an isolated singularity at z_σ of arbitrary high order, more precisely we will obtain

$$Du \cdot Dv = O(|x - z_\sigma|^{-2K}), \qquad \text{in} \quad \Omega,$$

with $K \geqslant n - 1$.

Now, if $\Lambda_a = \Lambda_b$, then we have

$$\int_\Omega (a - b)\, Du \cdot Dv = 0,$$

and letting $\sigma \to 0$, any continuous derivative of $(a - b)$ is forced to vanish on $\partial\Omega$.

Let us remark that Isakov already made use of singular solutions for the determination of discontinuities in the coefficient a from Λ_a [I], however, only Green's function type singularities were needed for his purpose.

Isolated singularities for elliptic equations have been widely studied in the past; see for instance [GS, B1, J, Ma]. The main objective of these studies has been the classification of singular solutions. However, for our purpose, the converse viewpoint, the one of existence, is necessary. The spirit of Theorem 1.1 below is the following.

Let $L = (\partial/\partial x_i)(a_{ij}(x)(\partial/\partial x_j))$ be an elliptic operator with sufficiently smooth coefficients and let h be any solution to $a_{ij}(x^0)(\partial^2 h/\partial x_i \partial x_j) = 0$ having an isolated singularity of finite order at x^0, then there exists a solution u to $Lu = 0$ with isolated singularity at x^0, which is asymptotic to h as $x \to x^0$.

A result of this kind has been proven by Marcus [Ma], but only for analytic coefficients, here instead we give a proof for Lipschitz continuous coefficients a_{ij} (in fact a little less suffices: $a_{ij} \in W^{1,p}$ with $p > n$, see Theorem 1.1).

Section 1 contains the statements of the main results.

In Section 2 we prove the results about the existence of singular solutions.

In Section 3 we prove the results on the inverse problem.

1. Statements of the Results

We consider elliptic operators

$$L = \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j}\right), \qquad \text{in} \quad B_R = \{x \in \mathbb{R}^n \mid |x| < R\}, \tag{1.1}$$

where the symmetric coefficient matrix $(a_{ij}(x))$ satisfies

$$\lambda^{-1} |\xi|^2 \leqslant a_{ij}(x)\, \xi_i \xi_j \leqslant \lambda |\xi|^2, \qquad \text{for every} \quad x, \xi, x \in B_R, \xi \in \mathbb{R}^n, \tag{1.2}$$

and

$$\| a_{ij} \|_{W^{1,p}(B_R)} \leqslant E, \qquad i, j = 1, ..., n, \tag{1.3}$$

here $p > n$ and λ, E are positive constants. Note that (1.3) implies that the coefficients a_{ij} are Hölder continuous with exponent $1 - n/p$.

For simplicity we will assume $a_{ij}(0) = \delta_{ij}$, in fact this situation can always be achieved by a linear change of variables.

Here and in the sequel, we will denote by D^k, k positive integer, the set of all derivatives of order k, in particular, $D =$ gradient, $D^2 =$ hessian matrix.

THEOREM 1.1 (Singular Solutions). *Let L satisfy* (1.1)–(1.3). *For every spherical harmonic S_m of degree $m = 0, 1, 2, ...,$ there exists $u \in W^{2,p}_{\text{loc}}(B_R \setminus \{0\})$ such that*

$$Lu = 0 \qquad in \quad B_R \setminus \{0\}, \tag{1.4}$$

and furthermore

$$\begin{aligned} u(x) &= \log |x|\, S_0\left(\frac{x}{|x|}\right) + w(x), && \textit{when} \quad n = 2 \textit{ and } m = 0, \\ u(x) &= |x|^{2-n-m}\, S_m\left(\frac{x}{|x|}\right) + w(x), && \textit{otherwise}, \end{aligned} \tag{1.5a}$$

where w satisfies

$$|w(x)| + |x|\,|Dw(x)| \leqslant C\,|x|^{2-n-m+\alpha}, \qquad in \quad B_R \setminus \{0\}, \tag{1.5b}$$

$$\left(\int_{r < |x| < 2r} |D^2 w|^p\right)^{1/p} \leqslant C r^{-n-m+\alpha+n/p}, \qquad \textit{for every} \quad r, 0 < r < R/2. \tag{1.5c}$$

Here α is any number such that $0 < \alpha < 1 - n/p$, and C is a constant depending only on α, n, p, R, λ, and E.

Remark 1.1. Note that if the hypothesis $a_{ij}(0) = \delta_{ij}$ is removed, then (1.5a) must be replaced by

$$\begin{aligned} u(x) &= \log |Jx|\, S_0\left(\frac{Jx}{|Jx|}\right) + w(x), && \text{when} \quad n = 2 \text{ and } m = 0, \\ u(x) &= |Jx|^{2-n-m}\, S_m\left(\frac{Jx}{|Jx|}\right) + w(x), && \text{otherwise}, \end{aligned} \tag{1.5a$'$}$$

where J is a symmetric matrix such that $J = \sqrt{(a_{ij}(0))^{-1}}$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$, and we denote by $\nu(x)$ the exterior unit normal vector to $\partial\Omega$ which exists for almost all $x \in \partial\Omega$.

We denote by $\Omega_r = \{x \in \bar{\Omega} \mid \operatorname{dist}(x, \partial\Omega) < r\}$ the r-neighborhood of $\partial\Omega$ in Ω.

For any positive $a \in L^\infty(\Omega)$ we define $\Lambda_a \colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by the formula

$$\langle \Lambda_a u, \varphi \rangle = \int_\Omega a\, Du \cdot D\varphi, \tag{1.6a}$$

where φ is any $H^1(\Omega)$ function and u is any $H^1(\Omega)$ solution to

$$\int_\Omega a\, Du \cdot D\psi = 0, \qquad \text{for every} \quad \psi \in H_0^1(\Omega). \tag{1.6b}$$

Note that, if $u \in C^1(\bar{\Omega})$, then $\Lambda_a u = a(\partial/\partial\nu)\, u$, almost everywhere on $\partial\Omega$. Note also that (1.6) implies that Λ_a is selfadjoint.

THEOREM 1.2 (Stability at the Boundary). *Let a, b satisfy*

$$0 < \lambda^{-1} \leqslant a(x), b(x) \leqslant \lambda, \qquad \textit{for every} \quad x \in \Omega, \tag{1.7}$$

$$\| a \|_{W^{1,p}(\Omega)}, \| b \|_{W^{1,p}(\Omega)} \leqslant E, \qquad \textit{for some} \quad p > n, \tag{1.8}$$

then we have

$$\| a - b \|_{L^\infty(\partial\Omega)} \leqslant C_1 \, \| \Lambda_a - \Lambda_b \|_{L(H^{1/2}, H^{-1/2})}. \tag{1.9}$$

Furthermore, if, for some integer $k \geqslant 1$ and some α, $0 < \alpha < 1$,

$$\| a - b \|_{C^{k+\alpha}(\bar{\Omega}_r)} \leqslant E_k, \tag{1.10}$$

then, setting

$$\delta_k = \prod_{j=0}^{k} \frac{\alpha}{\alpha + j},$$

we have

$$\| D^k(a - b) \|_{L^\infty(\partial\Omega)} \leqslant C_2 \, \| \Lambda_a - \Lambda_b \|_{L(H^{1/2}, H^{-1/2})}^{\delta_k}. \tag{1.11}$$

Here C_1 depends only on n, p, Ω, λ, and E, while C_2 depends only on $\alpha, k, r, n, p, \Omega, \lambda, E$, and E_k.

THEOREM 1.3 (Uniqueness at the Boundary). *Let a, b satisfy* (1.7), (1.8).

Suppose that $a-b \in C^k(\bar{U})$, *where* $U \subset \bar{\Omega}$ *is a neighborhood of some point* $y \in \partial\Omega$. *Then* $\Lambda_a = \Lambda_b$ *implies*

$$D^j a = D^j b \qquad \text{on} \quad \partial\Omega \cap \bar{U}, \qquad \text{for all} \quad j \leqslant k. \tag{1.12}$$

COROLLARY 1.1 (Piecewise Analytic Perturbations). *Let* a, b *satisfy* (1.7), (1.8) *with* $p = \infty$ (*i.e.*, $a, b \in \mathrm{Lip}(\bar{\Omega})$). *Suppose that* Ω *can be partitioned into a finite number of Lipschitz domains*, $\{A_j\}_{j=1,\dots,N}$, *such that* $a-b$ *is analytic on each* $\bar{A}_j$. *Then* $\Lambda_a = \Lambda_b$ *implies* $a = b$ *in all of* Ω.

COROLLARY 1.2 (Global Uniqueness and Stability). *Let* $n \geqslant 3$, *let* a, b *satisfy* (1.7), *and also*

$$\|a\|_{W^{2,\infty}(\Omega)}, \|b\|_{W^{2,\infty}(\Omega)} \leqslant E,$$

then we have

$$\|a-b\|_{L^\infty(\partial\Omega)} \leqslant \omega(\|\Lambda_a - \Lambda_b\|_{L(H^{1/2}, H^{-1/2})}),$$

where the function ω *satisfies*

$$\omega(t) \leqslant C\,|\log t|^{-\delta}, \qquad \text{for every} \quad t \in (0, 1),$$

and C *is a constant depending only on* n, Ω, λ, *and* E, *and* $\delta \in (0, 1)$ *depends only on* n.

Remark 1.2. In a private communication, C. Kenig has kindly pointed out to the author that he and D. Jerison have proved a result which seems to yield the global uniqueness when $n \geqslant 3$ and $a \in W^{2,p}(\Omega)$, $p > n/2$. Their argument is based on estimates due to Kenig, Ruiz, and Sogge [KRS]. See also [C].

Let us recall that the original global uniqueness result by Sylvester and Uhlmann was proven for $a \in C^\infty(\bar{\Omega})$. The stability result has been previously proven for coefficients having bounded $H^m(\Omega)$-norm, with $m > 2 + n/2$ [A].

We define the Dirichlet to Neumann operator Λ_A associated to the anisotropic equation div(A grad u) = 0, in the same way as in (1.6) by just replacing the scalar coefficient a by the matrix A. Note that if A is symmetric than Λ_A is selfadjoint.

We consider a differentiable, $n \times n$ symmetric matrix valued function $[\lambda^{-1}, \lambda] \ni t \to A(t)$, satisfying the conditions

$$\lambda^{-1}|\xi|^2 \leqslant A(t)\,\xi \cdot \xi \leqslant \lambda\,|\xi|^2, \qquad \text{for every} \quad \xi, t, \xi \in \mathbb{R}^n, t \in [\lambda^{-1}, \lambda], \tag{1.13}$$

$$|A'(t)| \leqslant E, \qquad \text{for every} \quad t \in [\lambda^{-1}, \lambda], \tag{1.14}$$

$$A'(t)\,\xi \cdot \xi \geqslant E^{-1}|\xi|^2, \qquad \text{for every} \quad \xi, t, \xi \in \mathbb{R}^n, t \in [\lambda^{-1}, \lambda]. \tag{1.15}$$

THEOREM 1.4 (Anisotropic Coefficients). *Let a, b satisfy* (1.7), (1.8) *and let* $A(t)$ *satisfy* (1.13)–(1.15). *We have the following results.*

(I) *The following estimate holds*:

$$\| A(a) - A(b)\|_{L^\infty(\partial\Omega)} \leqslant C_1 \, \| \Lambda_{A(a)} - \Lambda_{A(b)} \|_{L(H^{1/2}, H^{-1/2})}, \tag{1.16}$$

where C_1 *is as in Theorem* 1.2.

(II) *If, for some* $k \geqslant 1$,

$$\| A \|_{C^k([\lambda^{-1}, \lambda])} \leqslant E_k, \tag{1.17a}$$

$$\| a - b \|_{C^{k+\alpha}(\Omega_r)} \leqslant E_k, \tag{1.17b}$$

then we have

$$\| D^k(A(a) - A(b))\|_{L^\infty(\partial\Omega)} \leqslant C_2 \, \| \Lambda_{A(a)} - \Lambda_{A(b)} \|^{\delta_k}_{L(H^{1/2}, H^{-1/2})}, \tag{1.18}$$

where δ_k, C_2 *are as in Theorem* 1.2.

(III) *If* $A \in C^k([\lambda^{-1}, \lambda])$, *and* $a - b \in C^k(\bar{U})$, *where* $U \subset \bar{\Omega}$ *is a neighborhood of* $y \in \partial\Omega$, *then* $\Lambda_{A(a)} = \Lambda_{A(b)}$ *implies*

$$D^j A(a) = D^j A(b) \qquad \text{on} \quad \partial\Omega \cap \bar{U}, \qquad \text{for all} \quad j \leqslant k. \tag{1.19}$$

(IV) *Suppose that* $A \in C^\infty([\lambda^{-1}, \lambda])$, *a, b satisfy* (1.8) *with* $p = \infty$, *and* $a - b$ *is piecewise analytic in* Ω (*in the sense of Corollary* 1.1), *then* $\Lambda_{A(a)} = \Lambda_{A(b)}$ *implies* $A(a) = A(b)$ *in all of* Ω.

2. EXISTENCE OF SINGULAR SOLUTIONS

We start with three lemmas. We consider the case $n \geqslant 3$. The case $n = 2$ could be treated with minor modifications, or by specific two dimensional methods, like the theory of pseudo-analytic functions; see for instance [B2].

LEMMA 2.1. *Let* $u \in W^{2,p}_{\text{loc}}(B_R \setminus \{0\})$, $p > n$, *be such that, for some positive s*,

$$|u(x)| \leqslant |x|^{2-s}, \qquad \text{for every} \quad x \in B_R \setminus \{0\}, \tag{2.1}$$

$$\left(\int_{r < |x| < 2r} |Lu|^p \right)^{1/p} \leqslant A r^{n/p - s}, \qquad \text{for every} \quad r, 0 < r < R/2. \tag{2.2}$$

Then we have

$$|Du(x)| \leqslant C\,|x|^{1-s}, \qquad \textit{for every} \quad x \in B_R \setminus \{0\}, \tag{2.3a}$$

$$\left(\int_{r<|x|<2r} |D^2u|^p\right)^{1/p} \leqslant C r^{n/p-s}, \qquad \textit{for every} \quad r, 0<r<R/4. \tag{2.3b}$$

Here C depends only on A, n, p, λ, and E.

Proof. Straightforward consequence of interior Schauder estimates in L^p:

$$\left(\int_{r<|x|<2r} |D^2u|^p\right)^{1/p} + r^{1+n/p}\left(\sup_{r<|x|\leqslant 2r} |Du(x)|\right)$$
$$\leqslant C\left(\left(\int_{r/2<|x|<4r} |Lu|^p\right)^{1/p} + r^{-2}\left(\int_{r/2<|x|<4r} |u|^p\right)^{1/p}\right), \quad 0<r<R/4.$$

See for instance [N, GT]. ∎

LEMMA 2.2. *Let $f \in L^p_{\mathrm{loc}}(B_R \setminus \{0\})$ satisfy*

$$\left(\int_{r<|x|<2r} |f|^p\right)^{1/p} \leqslant A r^{n/p-s}, \qquad \textit{for every} \quad r, 0<r<R/2, \tag{2.4}$$

with $2<s<n<p$. Then there exists $u \in W^{2,p}_{\mathrm{loc}}(B_R \setminus \{0\})$ satisfying

$$Lu = f, \qquad \textit{in} \quad B_R \setminus \{0\}, \tag{2.5}$$

and

$$|u(x)| \leqslant C\,|x|^{2-s}, \qquad \textit{for every} \quad x \in B_R \setminus \{0\}, \tag{2.6}$$

where C depends only on A, s, n, p, R, λ, and E.

Proof. Let us assume in addition $f \in L^\infty(B_R)$. We consider

$$u(x) = \int_{B_R} G(x,y) f(y)\,dy, \tag{2.7}$$

where G is the Green function associated to L in B_R. Clearly u satisfies (2.5). Now we prove (2.6). As is well known (see, e.g., [Mi, Chap. IV]) we have $|G(x,y)| \leqslant C(n,\lambda)|x-y|^{2-n}$, for every $x \neq y$. Hence

$$|u(x)| \leqslant C[I_1 + I_2],$$

where

$$I_1 = \int_{|y| < |x|/2} |x-y|^{2-n} |f(y)|\, dy, \tag{2.8a}$$

$$I_2 = \int_{|x|/2 < |y| < R} |x-y|^{2-n} |f(y)|\, dy. \tag{2.8b}$$

Here, and in the sequel, we will make use of the following observation: if f satisfies (2.4), then, by Hölder's inequality, we have

$$\int_{r < |y| < 2r} |y|^{\mu} |f(y)|\, dy \leqslant C(\mu, s, n, A) \int_{r < |y| < 2r} |y|^{\mu - s}\, dy. \tag{2.9}$$

Now if $|y| < |x|/2$, then $|x-y| \geqslant |x|/2$, hence

$$\begin{aligned} I_1 &\leqslant C|x|^{2-n} \int_{|y| < |x|/2} |f(y)|\, dy \\ &= C|x|^{2-n} \sum_{k=1}^{\infty} \int_{2^{-k-1}|x| < |y| < 2^{-k}|x|} |f(y)| dy \\ &\leqslant C|x|^{2-n} \int_{|y| < |x|/2} |y|^{-s}\, dy \leqslant C|x|^{2-s}. \end{aligned}$$

Note that if we continue f to 0 outside of B_R, then (2.4) holds also for $r \geqslant R/2$. Therefore we have

$$\begin{aligned} I_2 &\leqslant \int_{|x|/2 < |y| < 2|x|} |x-y|^{2-n} |f(y)|\, dy \\ &\quad + C \sum_{k=1}^{\infty} \int_{2^k|x| < |y| < 2^{k+1}|x|} |x-y|^{2-n} |f(y)|\, dy \\ &\leqslant \left(\int_{|x|/2 < |y| < 2|x|} |x-y|^{(2-n)p'}\, dy \right)^{1/p'} \left(\int_{|x|/2 < |y| < 2|x|} |f(y)|^p\, dy \right)^{1/p} \\ &\quad + C \sum_{k=1}^{\infty} \int_{2^k|x| < |y| < 2^{k+1}|x|} |y|^{2-n} |f(y)|\, dy, \end{aligned}$$

and here $p' = p/(p-1)$. Therefore by (2.4), (2.9)

$$\begin{aligned} I_2 &\leqslant \left(\int_{|x|/2 < |y| < 2|x|} |x-y|^{(2-n)p'}\, dy \right)^{1/p'} C|x|^{n/p - s} \\ &\quad + C\left(\int_{|y| > 2|x|} |y|^{2-n-s}\, dy \right) \leqslant C|x|^{2-s}. \end{aligned} \tag{2.10}$$

Hence (2.6) follows. It remains to remove the extra hypothesis $f \in L^{\infty}(B_R)$. Let us set

$$f_N = \begin{cases} N, & \text{when } f > N, \\ f, & \text{when } |f| \leqslant N, \\ -N, & \text{when } f < -N. \end{cases}$$

Let u_N be the corresponding function obtained by (2.7). Now, each u_N satisfies $Lu_N = f_N$ in B_R, and the estimate (2.6) with C independent of N. Therefore, by interior L^p-Schauder estimates $\{u_N\}$ is uniformly bounded in $W^{2,p}_{\text{loc}}(B_R \setminus \{0\})$. We take u as the limit of a weakly converging subsequence of $\{u_N\}$. Such a limit satisfies both (2.5) and (2.6). ∎

LEMMA 2.3. *Let $s > n$ be a non-integral real number. Let f satisfy* (2.4) *with $p > n$. Then there exists $u \in W^{2,p}_{\text{loc}}(B_R \setminus \{0\})$ such that $\Delta u = f$ in $B_R \setminus \{0\}$ and* (2.6) *holds with C depending only on A, s, n, p, and R.*

Proof. We rephrase arguments in [B1, Ma]. Let $\Gamma(x-y) = -c_n |x-y|^{2-n}$ be the fundamental solution for the Laplace operator in $\mathbb{R}^n$. We see that, for $|y| < |x|$,

$$\Gamma(x-y) = -c_n \sum_{j=0}^{\infty} \frac{|y|^j}{|x|^{j+n-2}} C_j^{(n-2)/2}\left(\frac{y}{|y|} \cdot \frac{x}{|x|}\right), \tag{2.11}$$

where $C_j^{(n-2)/2}$ are Gegenbauer polynomials (see [E]). It can be shown that, for $|\rho| \leqslant 1$, $|C_j^{(n-2)/2}(\rho)| \leqslant \text{Const}\, j^{n-3}$, where the constant depends only on n (see for instance [CF; E, 315(13)]). We define, for $v = 0, 1, 2, \ldots,$

$$\Gamma_v(x, y) = \Gamma(x-y) + c_n \sum_{j=0}^{v} \frac{|y|^j}{|x|^{j+n-2}} C_j^{(n-2)/2}\left(\frac{y}{|y|} \cdot \frac{x}{|x|}\right).$$

Note that

$$\Delta_x \Gamma_v(x, y) = \delta(x-y), \qquad \text{for} \quad x \neq 0. \tag{2.12}$$

Recalling the cut-off argument exploited in Lemma 1.2 we can assume in addition $f \in L^{\infty}(B_R)$. We set

$$u(x) = \int_{B_R} \Gamma_v(x, y) f(y)\, dy, \tag{2.13}$$

where $v = [s] - n$. By (2.12) we have $\Delta u = f$ in $B_R \setminus \{0\}$. In order to obtain (2.6) we split the integral in (2.13) as

$$|u(x)| \leqslant C(I_2 + I_3 + I_4),$$

where I_2 is the same as in (2.8) and satisfies (2.10), while the terms I_3, I_4 are given by

$$I_3 = \sum_{j=0}^{\nu} j^{n-3} \left(\int_{|x|/2 < |y| < R} \frac{|y|^j}{|x|^{j+n-2}} |f(y)|\, dy \right),$$

$$I_4 = \sum_{j=\nu+1}^{\infty} j^{n-3} \left(\int_{|y| < |x|/2} \frac{|y|^j}{|x|^{j+n-2}} |f(y)|\, dy \right).$$

As in Lemma 2.2 we can continue f to 0 outside B_R and hence we have, by (2.9),

$$\begin{aligned} I_3 &\leqslant C \sum_{j=0}^{\nu} j^{n-3} \sum_{k=0}^{\infty} \left(\int_{2^{k-1}|x| < |y| < 2^k |x|} \frac{|y|^{j-s}}{|x|^{j+n-2}}\, dy \right) \\ &\leqslant C \sum_{j=0}^{\nu} j^{n-3} |x|^{2-n-j} \left(\int_{|x|/2 < |y|} |y|^{j-s}\, dy \right) \leqslant C |x|^{2-s}, \\ I_4 &\leqslant C \sum_{j=\nu+1}^{\infty} j^{n-3} \left(\sum_{k=1}^{\infty} |x|^{2-n-j} \int_{2^{-k-1}|x| < |y| < 2^{-k}|x|} |y|^{j-s}\, dy \right) \\ &\leqslant C \sum_{j=\nu+1}^{\infty} j^{n-3} |x|^{2-n-j} \left(\int_{|y| < |x|/2} |y|^{j-s}\, dy \right) \leqslant C |x|^{2-s}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 1.1. Consider, in $B_R \setminus \{0\}$,

$$H(x) = |x|^{2-n-m} S_m \left(\frac{x}{|x|} \right).$$

We seek w satisfying (1.5) and also

$$Lw = -LH, \qquad \text{in} \quad B_R \setminus \{0\}.$$

Now note

$$-LH = (\Delta - L) H = (\delta_{ij} - a_{ij}) \frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial a_{ij}}{\partial x_i} \frac{\partial H}{\partial x_j}.$$

Therefore, setting $\beta = 1 - n/p$, we have

$$\left(\int_{r < |x| < 2r} |LH|^p \right)^{1/p} \leqslant C(E, R)\, r^{\beta - m - n}.$$

Take α be any irrational number, $0 < \alpha < \beta$, and set, for $K = [m/\alpha] + 1$,

$$w = \left(\sum_{j=0}^{K-1} w_j \right) + W_K,$$

where w_0 is the solution to $\Delta w_0 = f$ given by Lemma 1.3 when $f = -LH$, and, inductively, w_j is the solution to $\Delta w_j = f$ given by Lemma 1.3 with $f = (\Delta - L)\, w_{j-1}$, $j = 1, \ldots, K-1$. Note that we obtain, for every $j = 0, \ldots, K-1$,

$$|w_j| \leqslant C\,|x|^{2-n-m+(j+1)\alpha},$$

$$\left(\int_{r<|x|<2r} |(\Delta - L)\, w_j|^p\right)^{1/p} \leqslant C r^{n/p-n-m+(j+1)\alpha}.$$

Consequently, we define W_K as the solution given by Lemma 2.2 to $LW_K = f$ with $f = (\Delta - L)\, w_{K-1}$. Such W_K satisfies

$$|W_K(x)| \leqslant C\,|x|^{2-n-m+(K+1)\alpha} \leqslant C\,|x|^{2-n+\alpha}.$$

Finally, we have

$$\begin{aligned} Lw &= \left(\sum_{j=0}^{K-1} Lw_j\right) + LW_K \\ &= \left(\sum_{j=0}^{K-1} \Delta w_j\right) - \left(\sum_{j=0}^{K-1} (\Delta - L)\, w_j\right) + LW_K = -LH, \qquad \text{in} \quad B_R\setminus\{0\}, \end{aligned}$$

and also

$$|w(x)| \leqslant C\,|x|^{2-n-m+\alpha}.$$

The estimates on Dw, D^2w follow from Lemma 2.1. ∎

3. Stability and Uniqueness

We have assumed that $\partial\Omega$ is Lipschitz, and hence the normal vector field ν may be discontinuous on $\partial\Omega$. It is convenient to introduce a new unitary vector field $\tilde{\nu}$ defined on a neighborhood of $\partial\Omega$ such that: (i) $\tilde{\nu}$ is C^∞ smooth, (ii) $\tilde{\nu}$ is non-tangential to $\partial\Omega$, that is, for almost every $x \in \partial\Omega$, $\tilde{\nu}(x)\cdot\nu(x) \geqslant C > 0$, where the constant C depends only on Ω.

For any $x^0 \in \partial\Omega$, let us set $z_\sigma = x^0 + \sigma\tilde{\nu}(x^0)$, and we will have

$$C\sigma \leqslant d(z_\sigma, \partial\Omega) \leqslant \sigma, \qquad \text{for every} \quad \sigma, 0 \leqslant \sigma \leqslant \sigma^0,$$

where σ^0 and C depend only on Ω.

Now, fixing $R > 2 \operatorname{diam} \Omega$, we can continue a, b to $B_R(z_\sigma)$ in such a way that

$$0 < \bar{\lambda}^{-1} \leqslant a, b \leqslant \bar{\lambda}, \qquad \text{in} \quad B_R(z_\sigma),$$

$$\|a\|_{W^{1,p}(B_R(z_\sigma))}, \|b\|_{W^{1,p}(B_R(z_\sigma))} \leqslant \bar{E},$$

where $\bar{\lambda}$, $\bar{E}$ depend only on λ, E, Ω, and R.

The following lemma is a consequence of Theorem 1.1.

LEMMA 3.1. *Let a, b satisfy* (1.7), (1.8). *For every non-negative integer m there exist solutions u, $v \in W^{2,p}(\Omega)$ to*

$$\operatorname{div}(a \operatorname{grad} u) = \operatorname{div}(b \operatorname{grad} v) = 0, \qquad in \quad \Omega, \tag{3.1}$$

satisfying

$$|Du|, |Dv| \leqslant C\,|x - z_\sigma|^{1-n-m}, \quad for\ every \quad x \in \Omega, \tag{3.2a}$$

$$Du \cdot Dv \geqslant |x - z_\sigma|^{2-2(n+m)}, \qquad for\ every \quad x \in \Omega,\ such\ that\ |x - z_\sigma| \leqslant r_0. \tag{3.2b}$$

Here r_0 and C are constants depending only on λ, E, Ω, n, p, and m.

Proof. For simplicity we can set $z_\sigma = 0$. By Theorem 1.1 it suffices to find a spherical harmonic S_m such that

$$\left| D\left(|x|^{2-n-m} S_m\left(\frac{x}{|x|}\right)\right)\right|^2 \geqslant 2\,|x|^{2-2(n+m)}.$$

This is a trivial task when $n = 2$. Let $n \geqslant 3$ and choose $S_m(x/|x|) = A C_m^{(n-2)/2}(x_n/|x|)$, $A = \text{const} \neq 0$, where $C_m^{(n-2)/2}(t)$ are the same Gegenbauer polynomials appearing in (2.11).

Note that formula (2.11) can be used to show that $|x|^{2-n-m} C_m^{(n-2)/2}(x_n/|x|)$ is a homogeneous harmonic function of degree $2-n-m$. Setting $t = x_n/|x|$ we have

$$\left| D\left(|x|^{2-n-m} S_m\left(\frac{x}{|x|}\right)\right)\right|^2 = A^2\,|x|^{2-2(n+m)} \left((2-n-m)^2 (C_m^{(n-2)/2}(t))^2 + \left(\frac{d}{dt} C_m^{(n-2)/2}(t)\right)^2\right).$$

We must prove that $C_m^{(n-2)/2}(t)$ and $(d/dt)\, C_m^{(n-2)/2}(t)$ cannot vanish simultaneously for any t, $|t| \leqslant 1$. Let us notice that $C_m^{(n-2)/2}(\pm 1) \neq 0$ and also that $C_m^{(n-2)/2}$ solves the equation

$$(t^2 - 1)\, w'' + (n-1)\, w' - m(m+n-2)\, w = 0$$

(see [E, 3.15(7) and 3.15(21)]). Now, by the Cauchy uniqueness theorem, for any t, $|t| < 1$, we have $(d/dt)\, C_m^{(n-2)/2}(t) \neq 0$ when $C_m^{(n-2)/2}(t) = 0$. ∎

In the sequel we will make use of the following inequality. A proof will be given at the end of this section.

LEMMA 3.2. *For any $f \in C^{1+\alpha}(\bar{\Omega})$, $0 < \alpha \leqslant 1$, we have*

$$\|Df\|_{L^\infty(\partial\Omega)} \leqslant C(\Omega) \left\{ \left\| \frac{\partial}{\partial \tilde{v}} f \right\|_{L^\infty(\partial\Omega)} + \|f\|_{L^\infty(\partial\Omega)}^{\alpha/(1+\alpha)}\, \|f\|_{C^{1+\alpha}(\bar{\Omega})}^{1/(1+\alpha)} \right\}. \tag{3.3}$$

Proof of Theorem 1.2. We start by proving (1.9). Let $x^0 \in \partial\Omega$ be such that $|(a-b)(x^0)| = \|a-b\|_{L^\infty(\partial\Omega)}$, and set, for convenience, $(a-b)(x^0) > 0$. We have, for every $x \in \bar{\Omega}$,

$$\|a-b\|_{L^\infty(\partial\Omega)} = (a-b)(x^0) \leqslant (a-b)(x) + C\,|x-x^0|^\beta, \qquad \beta = 1 - n/p.$$

Now we apply the formula (see (1.6) and the following remarks)

$$\int_\Omega (a-b)\, Du \cdot Dv = \langle (\Lambda_a - \Lambda_b)\, u, v \rangle, \tag{3.4}$$

which holds for all u, v satisfying (3.1). We choose the solutions u, v given by Lemma 3.1. Setting $\rho = r_0$ and using (3.2), we obtain, for any $\sigma \leqslant \rho/2$,

$$\begin{aligned} &\|a-b\|_{L^\infty(\partial\Omega)} \int_{B_\rho(z_\sigma)\cap\Omega} |x-z_\sigma|^{2-2(n+m)} \\ &\quad\leqslant C\Bigg(\int_{\Omega\setminus B_\rho(z_\sigma)} |a-b|\,|x-z_\sigma|^{2-2(n+m)} \\ &\qquad + \int_{B_\rho(z_\sigma)\cap\Omega} |x-x_0|^\beta\,|x-z_\sigma|^{2-2(n+m)} \\ &\qquad + \|\Lambda_a - \Lambda_b\|_{L(H^{1/2},\,H^{-1/2})}\,\|u\|_{H^{1/2}(\partial\Omega)}\,\|v\|_{H^{1/2}(\partial\Omega)}\Bigg). \end{aligned} \tag{3.5}$$

Note that, possibly adding suitable constants to u and v, we can assume

$$\int_\Omega u = \int_\Omega v = 0, \tag{3.6}$$

and, therefore, by (3.2a),

$$\begin{aligned} &\|u\|^2_{H^{1/2}(\partial\Omega)},\ \|v\|^2_{H^{1/2}(\partial\Omega)} \\ &\qquad \leqslant \begin{cases} C\,|\log \sigma/r_0|, & \text{when } n=2 \text{ and } m=0, \\ C\sigma^{2-n-2m}, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.7}$$

Now we have

$$\int_{B_\rho(z_\sigma)\cap\Omega} |x-z_\sigma|^{2-2(n+m)} \geqslant \begin{cases} C\,|\log \sigma/\rho|, & \text{when } n=2 \text{ and } m=0, \\ C\sigma^{2-n-2m}, & \text{otherwise,} \end{cases} \tag{3.8}$$

$$\int_{\Omega\setminus B_\rho(z_\sigma)} |x-z_\sigma|^{2-2(n+m)} \leqslant C, \tag{3.9}$$

and

$$\int_{B_\rho(z_\sigma)\cap\Omega} |x-z_\sigma|^{2-2(n+m)}\,|x-x_0|^\beta$$

$$\leqslant \begin{cases} C, & \text{when } n=2 \text{ and } m=0, \\ C\sigma^{2-n-2m+\beta}, & \text{otherwise.} \end{cases}$$

Hence, choosing for instance $m=0$, (3.5) yields

$$\|a-b\|_{L^\infty(\partial\Omega)} \leqslant C\{\omega(\sigma)+\|\Lambda_a-\Lambda_b\|_{L(H^{1/2},H^{-1/2})}\},$$

where $\omega(\sigma)$ is infinitesimal as $\sigma\to 0$, and (1.9) follows.

In order to prove (1.11) we will show that (1.10) implies

$$\left\|\frac{\partial^j}{\partial\tilde{v}^j}(a-b)\right\|_{L^\infty(\partial\Omega)} \leqslant C\,\|\Lambda_a-\Lambda_b\|^{\delta_j}_{L(H^{1/2},H^{-1/2})},$$

$$\delta_j=\prod_{i=0}^{j}\frac{\alpha}{\alpha+i}, \qquad \text{for every} \quad j, 0\leqslant j\leqslant k. \tag{3.10}$$

The estimate (1.11) will follow from (3.10) and an iterated use of (3.3). We proceed by induction on k. For $k=0$, (3.10) is given by (1.9). Without loss of generality, we can assume that there exists $x^0\in\partial\Omega$ such that

$$(-1)^k\frac{\partial^k}{\partial\tilde{v}^k}(a-b)(x^0)=\left\|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)\right\|_{L^\infty(\partial\Omega)}.$$

Any $x\in\bar{\Omega}_{\sigma_0}$ can be uniquely represented as $x=y-s\tilde{v}(y)$, with $y\in\partial\Omega$ and $0\leqslant s\leqslant\sigma_0$. Note also $Cs\leqslant d(x,\partial\Omega)\leqslant s$, $|y-x^0|\leqslant C\,|x-x^0|$. Hence, for every $s\leqslant\min\{\sigma_0,r\}$, we have, by (1.10),

$$\left|(a-b)(x)-\sum_{j=0}^{k-1}\frac{\partial^j(a-b)}{\partial\tilde{v}^j}(y)\frac{(-s)^j}{j!}-\frac{\partial^k(a-b)}{\partial\tilde{v}^k}(x^0)\frac{(-s)^k}{k!}\right|$$
$$\leqslant Cs^k\,|x-x^0|^\alpha,$$

and therefore

$$\left\|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)\right\|_{L^\infty(\partial\Omega)}s^k$$
$$\leqslant k!(a-b)(x)+C\left\{\sum_{j=0}^{k-1}\left\|\frac{\partial^j}{\partial\tilde{v}^j}(a-b)\right\|_{L^\infty(\partial\Omega)}s^j+s^k\,|x-x^0|^\alpha\right\}. \tag{3.11}$$

Hence using again formula (3.4) and the solutions u, v given by Lemma 3.1, we obtain for $\rho=\min\{\sigma_0, r, r_0\}$, $\sigma\leqslant\rho/2$,

$$\begin{aligned}
&\left\|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)\right\|_{L^\infty(\partial\Omega)}\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^k\\
&\quad\leqslant C\left(\int_{\Omega\setminus B_\rho(z_\sigma)}|a-b|\,|x-z_\sigma|^{2-2(n+m)}\right.\\
&\qquad\left.+\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^k\,|x-x_0|^\alpha\right)\\
&\qquad+C\sum_{j=0}^{k-1}\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^j\left\|\frac{\partial^j}{\partial\tilde{v}^j}(a-b)\right\|_{L^\infty(\partial\Omega)}\\
&\qquad+C\,\|\Lambda_a-\Lambda_b\|_{L(H^{1/2},H^{-1/2})}\,\|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)}.
\end{aligned}\tag{3.12}$$

Now we see that, for every $m>(k-1)/2$,

$$\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^k\geqslant C\sigma^{2-n+2m+k},\tag{3.13}$$

and furthermore

$$\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^j\leqslant C\sigma^{2-n+2m+j},\qquad j=0,\dots,k-1,$$

$$\int_{B_\rho(z_\sigma)\cap\Omega}|x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^k\,|x-x_0|^\alpha\leqslant C\sigma^{2-n+2m+k+\alpha}.$$

Recalling (3.7), (3.9), and the induction hypothesis, we have

$$\begin{aligned}
&\left\|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)\right\|_{L^\infty(\partial\Omega)}\\
&\quad\leqslant C\left\{\sum_{j=0}^{k-1}\|\Lambda_a-\Lambda_b\|^{\delta_j}_{L(H^{1/2},H^{-1/2})}\sigma^{j-k}+\sigma^\alpha+\sigma^{-2+n+2m-k}\right\}.
\end{aligned}$$

Now, choosing m large,

$$\left\|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)\right\|_{L^\infty(\partial\Omega)}\leqslant C\left\{\|\Lambda_a-\Lambda_b\|^{\delta_{k-1}}_{L(H^{1/2},H^{-1/2})}\sigma^{-k}+\sigma^\alpha\right\},\tag{3.14}$$

and optimizing with respect to σ we obtain (3.10). ▮

Proof of Theorem 1.3. It suffices to prove

$$\frac{\partial^j}{\partial \tilde{v}^j}(a-b)=0 \qquad \text{on} \quad \partial\Omega \cap U, \qquad \text{for all} \quad j \leqslant k,$$

and we can proceed by induction on k. By contradiction let $x^0 \in \partial\Omega \cap U$ be such that $(-1)^k (\partial^k/\partial\tilde{v}^k)(a-b)(x^0)>0$. We can find $\rho>0$ such that for every $x \in \Omega \cap B_\rho(z_\sigma)$ we have $x=y-s\tilde{v}(y)$, $y \in \partial\Omega$, and

$$(a-b)(x) \geqslant \frac{1}{2}\frac{(-s)^k}{k!}\frac{\partial^k}{\partial\tilde{v}^k}(a-b)(x^0).$$

Rephrasing the arguments leading to (3.12) we obtain, by the induction hypothesis,

$$(-1)^k \frac{\partial^k}{\partial\tilde{v}^k}(a-b)(x^0)\int_{B_\rho(z_\sigma)\cap\Omega} |x-z_\sigma|^{2-2(n+m)}(d(x,\partial\Omega))^k$$
$$\leqslant C\int_{\Omega\setminus B_\rho(z_\sigma)} |x-z_\sigma|^{2-2(n+m)}.$$

Therefore by (3.13) we obtain

$$\left|\frac{\partial^k}{\partial\tilde{v}^k}(a-b)(x^0)\right| \leqslant C\sigma^{-2+n+2m-k},$$

which, for m large, is infinitesimal as $\sigma \to 0$. ∎

Proof of Corollary 1.1. We rephrase an argument due to Kohn and Vogelius [KV2]. We collect the A_j's into layers L_k by the contruction

$$\Omega_0=\Omega,$$
$$L_k=\bigcup\{A_j \subset \Omega_{k-1} \mid \partial A_j \cap \partial\Omega_{k-1} \text{ has non-empty interior in } \partial\Omega_{k-1}\}, \qquad k=1,2,\ldots,$$
$$\Omega_k=\Omega_{k-1}\setminus\bar{L}_k, \qquad k=1,2,\ldots.$$

Clearly $\bigcup L_k=\Omega$. We will prove iteratively that $a=b$ on L_k. On L_1, $a=b$ by Theorem 1.3 and by analytic continuation. Hence $\Lambda_a=\Lambda_b$ implies

$$\int_{\Omega_1}(a-b)\,Du\cdot Dv=0$$

for all u, v solving $\operatorname{div}(a \operatorname{grad} u)=\operatorname{div}(b \operatorname{grad} v)=0$ in Ω. Since a, b are Lipschitz continuous we can use the Runge approximation property (see [KV2, Mi, Chap. III, Sect. 19]) and deduce that the above identity holds for all solutions u, v in any neighborhood of Ω_1. Therefore we can make use of the singular solutions in Lemma 3.1 and again by Theorem 1.3, obtain $a=b$ in L_2. We can repeat the procedure for all L_k's. ▮

Proof of Corollary 1.2. This proof can be obtained as a straightforward adaptation of the method employed in [A] in combination with the results in [NSU] (especially Lemma 2.2, which replaces Lemma 2 in [A]) and Theorem 1.2 above (which replaces Proposition 2 in [A]). ▮

Proof of Theorem 1.4. We just sketch the proof of (I) and (II) in order to point out the modifications of the proofs of Theorems 1.2 and 1.3 and of Corollary 1.1 which are needed.

We start once more from the formula

$$\int_\Omega (A(a)-A(b))\, Du\cdot Dv = \langle (\Lambda_{A(a)}-\Lambda_{A(b)})\, u, v\rangle, \tag{3.15}$$

where u, v are two arbitrary solutions to $\operatorname{div}(A(a)\operatorname{grad} u)=\operatorname{div}(A(b)\operatorname{grad} v)=0$, respectively. As in Theorem 1.2 we choose x^0 such that $a(x^0)-b(x^0)=\|a-b\|_{L^\infty(\partial\Omega)}$ and $z_\sigma = x^0+\sigma\tilde{v}(x^0)$. We fix u, v having a Green function type singularity at z_σ:

$$Du(x) = |J_a(x-z_\sigma)|^{-n} J_a^2(x-z_\sigma) + O(|x-z_\sigma|^{1-n+\alpha}),$$
$$Dv(x) = |J_b(x-z_\sigma)|^{-n} J_b^2(x-z_\sigma) + O(|x-z_\sigma|^{1-n+\alpha}).$$

Here $J_a=\sqrt{(A(a(x^0)))^{-1}}$, $J_b=\sqrt{(A(b(x^0)))^{-1}}$. We can proceed as in the proof of Theorem 1.2 and obtain the following analogue to (3.5):

$$\begin{aligned}
&\int_{B_\rho(z_\sigma)\cap\Omega} [J_b^2(A(a(x^0))-A(b(x^0)))\, J_a^2(x-z_\sigma)]\cdot(x-z_\sigma) \\
&\qquad \times |J_a(x-z_\sigma)|^{-n}\, |J_b(x-z_\sigma)|^{-n} \\
&\qquad \leqslant C\left\{\int_{\Omega\setminus B_\rho(z_\sigma)} |A(a)-A(b)|\, |x-z_\sigma|^{2-2n}\right. \\
&\qquad \left. + \int_{B_\rho(z_\sigma)\cap\Omega} |x-x^0|^\beta\, |x-z_\sigma|^{2-2n}\right\} \\
&\qquad + C\, \|\Lambda_{A(a)}-\Lambda_{A(b)}\|_{L(H^{1/2}, H^{-1/2})}\, \|u\|_{H^{1/2}(\partial\Omega)}\, \|v\|_{H^{1/2}(\partial\Omega)}.
\end{aligned}$$

Here the right hand side can be estimated as before. Note that, by (1.13) and (1.15),

$$[J_b^2(A(a(x^0))-A(b(x^0)))\, J_a^2(x-z_\sigma)]\cdot(x-z_\sigma) \geqslant C(a(x^0)-b(x^0))|x-z_\sigma|^2,$$

where C depends only on n, λ, and E. We obtain, by (1.14),

$$\|A(a)-A(b)\|_{L^\infty(\partial\Omega)} \leqslant E\,\|a-b\|_{L^\infty(\partial\Omega)} \leqslant C\,\{\omega(\sigma)+\|\Lambda_{A(a)}-\Lambda_{A(b)}\|_{L(H^{1/2},H^{-1/2})}\},$$

where $\omega(\sigma)\to 0$ as $\sigma\to 0$, and (I) follows.

In order to prove (II) we show, by induction on k, that

$$\left\|\frac{\partial^j}{\partial\tilde{v}^j}(a-b)\right\|_{L^\infty(\partial\Omega)} \leqslant C\,\|\Lambda_{A(a)}-\Lambda_{A(b)}\|^{\delta_j}_{L(H^{1/2},H^{-1/2})}, \qquad \text{for every} \quad j, 0\leqslant j\leqslant k, \tag{3.16}$$

in fact the use of Lemma 3.2 and the C^k-smoothness of $A(t)$ yield (1.18). Considering x^0, z_σ as in the proof of Theorem 1.2, we choose

$$u(x)=|J_a(x-z_\sigma)|^{2-n-m}\,S_m\left(\frac{J_a(x-z_\sigma)}{|J_a(x-z_\sigma)|}\right)+O(|x-z_\sigma|^{2-n-m+\alpha}),$$

$$v(x)=|J_b(x-z_\sigma)|^{2-n-m}\,S_m\left(\frac{J_b(x-z_\sigma)}{|J_b(x-z_\sigma)|}\right)+O(|x-z_\sigma|^{2-n-m+\alpha}),$$

where $S_m(x/|x|)=C_m^{(n-2)/2}(x_n/|x|)$, as in the proof of Lemma 3.1, and J_a, J_b are as above. It is easily seen that

$$|Du-Dv|\leqslant C(|x-z_\sigma|^{1-n-m}\,|a(x^0)-b(x^0)|+|x-z_\sigma|^{1-n-m+\alpha}),$$

moreover we can use the mean value theorem

$$[(A(a)-A(b))\,Du]\cdot Du=(a-b)[A'(c)\,Du]\cdot Du,$$

$$c(x)=a(x)+t(x)(b(x)-a(x)), \qquad 0<t(x)<1,$$

and the monotonicity assumption (1.15). Hence we can rephrase the arguments leading to (3.14) and eventually obtain (3.16). ∎

Proof of Lemma 3.2. By the use of a partition of the unity on $\partial\Omega$ and of smooth changes of variables, we can reduce the proof of (3.3) to the proof of the inequality

$$\|Df\|_{L^\infty(\Gamma)}\leqslant C\left\{\left\|\frac{\partial}{\partial x_n}f\right\|_{L^\infty(\Gamma)}+\|f\|^{\alpha/(1+\alpha)}_{L^\infty(\Gamma)}\,\|f\|^{1/(1+\alpha)}_{C^{1+\alpha}(\bar{G})}\right\}, \tag{3.17}$$

where Γ and G are defined by

$$\Gamma=\{x\in\mathbb{R}^n\,|\,|x_i|\leqslant\rho,\ i=1,\dots,n-1,\ x_n=F(x')\},$$

$$G=\{x\in\mathbb{R}^n\,|\,|x_i|\leqslant\rho,\ i=1,\dots,n-1,\ 0<x_n<F(x')\},$$

and $F = F(x_1, \ldots, x_{n-1})$, $|x_i| < \rho$, $i = 1, \ldots, n-1$, is a positive Lipschitz continuous function.

We can further reduce the problem to the case $n = 2$. In fact (3.17) is easily deduced from

$$\left\| \frac{\partial f}{\partial x_i} \right\|_{L^r(\Gamma)} \leqslant C \left\{ \left\| \frac{\partial}{\partial x_n} f \right\|_{L^\infty(\Gamma)} + \| f \|_{L^\infty(\Gamma)}^{\alpha/(1+\alpha)} \times \left(\| f \|_{L^\infty(G)} + \left\| \frac{\partial f}{\partial x_i} \right\|_{C^\alpha(\bar{G})} + \left\| \frac{\partial f}{\partial x_n} \right\|_{C^\alpha(\bar{G})} \right)^{1/(\alpha+1)} \right\},$$

$$\text{for every} \quad i = 1, \ldots, n-1, \quad (3.18)$$

and here only two variables are relevant. So let us pose $n = 2$ and let us prove (3.18).

For any two points $(x_1, F(x_1)), (y_1, F(y_1)) \in \Gamma$, $x_1 \neq y_1$, we have

$$\begin{aligned} f(y_1, F(y_1)) &- f(x_1, F(x_1)) \\ &= \int_0^1 \frac{\partial}{\partial x_1} f(x_1 + t(y_1 - x_1), F(x_1 + t(y_1 + x_1)))(y_1 - x_1)\, dt \\ &\quad + \int_0^1 \frac{\partial}{\partial x_2} f(x_1 + t(y_1 - x_1), F(x_1 + t(y_1 - x_1))) \\ &\quad \times \frac{\partial}{\partial x_1} F(x_1 + t(y_1 - x_1))(y_1 - x_1)\, dt. \end{aligned}$$

Note that the first integrand is continuous with respect to t, therefore there exists $z_1 = x_1 + \tau(y_1 - x_1)$, $0 \leqslant \tau \leqslant 1$, such that

$$\left| \frac{\partial}{\partial x_1} f(z_1, F(z_1)) \right| \leqslant L \left\| \frac{\partial}{\partial x_2} f \right\|_{L^\infty(\Gamma)} + \frac{2}{|x_1 - y_1|} \| f \|_{L^\infty(\Gamma)},$$

here L is the Lipschitz constant for F. We obtain

$$\begin{aligned} &\left| \frac{\partial}{\partial x_1} f(x_1, F(x_1)) \right| \\ &\quad \leqslant C \left\{ \left\| \frac{\partial}{\partial x_2} f \right\|_{L^\infty(\Gamma)} + \frac{1}{|x_1 - y_1|} \| f \|_{L^\infty(\Gamma)} + |x_1 - y_1|^\alpha \| Df \|_{C^\alpha(\bar{G})} \right\}. \end{aligned}$$

By the arbitrariness of y_1 we can choose $|x_1 - y_1|$ to be any number h, $0 < h \leqslant \rho$, hence optimizing with respect to h, we obtain (3.18). ∎

REFERENCES

[A] G. ALESSANDRINI, Stable determination of conductivity by boundary measurements, *Appl. Anal.* **27** (1988), 153–172.

[B1] L. BERS, Local behavior of solutions of general linear elliptic equations, *Comm. Pure Appl. Math.* **8** (1955), 473–496.

[B2] L. BERS, An outline of pseudo-analytic functions, *Bull. Amer. Math. Soc.* **62** (1956), 291–331.

[CF] L. CAFFARELLI AND A. FRIEDMAN, Partial regularity of the zero-set of solutions of linear and superlinear elliptic equations, *J. Differential Equations* **60** (1985), 420–433.

[C] S. CHANILLO, A problem in electrical prospection and an n-dimensional Borg–Levinson theorem, preprint.

[E] A. ERDELY *et al.*, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.

[GS] D. GILBARG AND J. SERRIN, On isolated singularities of solutions pf second order elliptic differential equations, *J. Anal. Math.* **4** (1955), 309–340.

[GT] D. GILBARG AND N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order," 2nd ed., Springer-Verlag, Berlin, 1983.

[I] V. ISAKOV, On uniqueness of recovery of a discontinuous conductivity coefficient, *Comm. Pure Appl. Math.* **41** (1988), 865–878.

[J] F. JOHN, The fundamental solution of linear elliptic equations with analytic coefficients, *Comm. Pure Appl. Math.* **3** (1950), 273–304.

[KRS] C. KENIG, A. RUIZ, AND C. SOGGE, Sobolev inequalities and unique continuation for second order constant coefficient differential equations, *Duke Math. J.* **55** (1987), 329–348.

[KV1] R. KOHN AND M. VOGELIUS, Determining conductivity by boundary measurements, *Comm. Pure Appl. Math.* **37** (1984), 113–123.

[KV2] R. KOHN AND M. VOGELIUS, Determining conductivity by boundary measurements. II. Interior results, *Comm. Pure Appl. Math.* **38** (1985), 643–667.

[Ma] M. MARCUS, Local behaviour of singular solutions of elliptic equations, *Ann. Scuola Norm. Sup. Pisa* **19** (1965), 519–561.

[Mi] C. MIRANDA, "Partial Differential Equations of Elliptic Type," 2nd rev. ed., Springer-Verlag, Berlin, 1970.

[NSU] A. NACHMAN, J. SYLVESTER, AND G. UHLMANN, An n-dimensional Borg–Levinson theorem, *Comm. Math. Phys.* **115** (1988), 593–605.

[N] L. NIRENBERG, Estimates and existence of solutions of elliptic equations, *Comm. Pure Appl. Math.* **9** (1956), 509–530.

[SU1] J. SYLVESTER AND G. UHLMANN, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* **125** (1987), 153–169.

[SU2] J. SYLVESTER AND G. UHLMANN, Inverse boundary value problems at the boundary-continuous dependence, *Comm. Pure Appl. Math.* **41** (1988), 197–219.